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On Methods Derived from Hansen-Patrick Formula for Refining Zeros of Polynominal Equation

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Abstract. A one-parameter family of iteration functions as derived by Hansen and Patrick (1977) was studied. The Halley's method was of particular interest, which was modified by using the Taylor polynomial equation of order two to obtain the well-known Chevbyshev's iteration formula. Further, using the Laguerre's disk, two new methods were constructed out of the Chevbyshev's functional iteration fomula. The obtained methods may, and often will, depend on the already calculated values.

Keywords: Hansen-Patrick formula, binomial series, Taylor polynomial equation, Laguerre's disk, polynomial zeros, Chevbyshev's functional iteration

Introduction

The principal objective of this study was to draw attention to the family of one-parameter iteration formulae derived by Hansen and Patrick (1977) for finding zeros of polynomial equation.

$$P(z) = 0 \tag{1.1}$$

It was assumed that P was real or complex, and possessed a certain number of derivatives necessary in the neighbourhood zeros of p. For convenience, p was specified to be the function of z with simple zero, ζ.

Then for $h(\zeta) \neq 0$

where h was the reduced polynomial of p

the equation (1.1) will thus assume the form:

$$P = (z - \zeta)\, h \tag{1.2}$$

By taking log of both sides of (1.2) and differentiating the resulting expression with respect to z, the following was obtained:

$$\frac{P'}{P} = \frac{1}{z - \zeta} + \frac{h'}{h} \tag{1.3}$$

Continuing, after some serious but rigorous analysis, Hansen and Patrick (1977) obtained a family of functional iterative formulae in the form:

$$z_i^{(k+1)} = \Phi(z_i^{(k)}) \quad (k = 0, 1, ...) \tag{1.4}$$

where Φ is a rational map given as:

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$$\Phi(z_i^{(k)}) = z_i^{(k)} - \frac{(\alpha + 1)p}{\alpha\, p' \pm [(p')^2\, (\alpha + 1)pp'']^{1/2}} \tag{1.5}$$

It is remarked that α appearing in (1.5) is a variable parameter that rules the governing equation (1.4), which is based on the approximation of second order derivative of h to the square of its first derivative.

Interest was motivated by organizing the remaining parts of the present study in the Results and Discussion section as follows.

a. Some cubically convergent methods that could be obtained when the values of α were in the region of (-1.1) was investigated as a class of Hansen-Patrik iterative formulae. This was done in particular, by neglecting some order of approximations higher than the term pp'' appearing in the denominator parts of these methods, and if it was assumed further that $|p|$ was sufficiently small in magnitude, then a limiting case of Halley's iterative formula of third order was obtained.

One disadvantage of Laguerre's method, as well as the Euler and Ostrowski formulae, is that they may occasionally branch off into complex plane even if the roots of the polynomials are real.

b. A functional iterative method was derived from the substitution of Halley's correction formula into Taylor polynomial of order two for a function p. The obtained method was familiar to the third order convergent Chevbyshev's formula. More useful information may be seen from Jarratt (1968).

In a second approach, the p'' appearing in the Chevbyshev's formula was approximated by the first finite difference approximation wherein the bound due to Laguerre was used (Braess and Hadeler, 1973). In this way, new iterative methods were obtained. The obtained methods may, and often will, depend on the already calculated values.

c. Finally, a sample numerical example was illustrated with these methods and the results so obtained were noted to be quite accurate as the solutions were approximated within 10^{-5} in the infinity norm.

Results and Discussion

a. A class of Hansen-Patrick iterative formulae. The values of α are crucial factors in establishing a family of methods that are iterative in nature for the determination of methods (Hansen and Patrick, 1977). For example, for $\alpha = 0$, a method due to Ostrowski (1966) was obtained from (1.4) in the form:

$$z_i^{(k+1)} = z_i^{(k)} - \frac{p}{\pm[(p')^2 - p'']^{1/2}} \qquad (2.1)$$

(k = 0, 1,.......)

By letting $\alpha = \infty$, a limiting case of the class of methods (1.4) is the Newton's second order method:

$$z_i^{(k+1)} = z_i^{(k)} - \frac{p}{p'} \qquad (2.2)$$

If $\alpha = 1$, a method due to Euler is obtained in the form:

$$z_i^{(k+1)} = z_i^{(k)} - \frac{2p}{p' + [(p')^2 - 2pp'']^{1/2}}$$

(k = 0, 1,.......) (2.3)

In the case of $\alpha = -1$, after some minor rearrangements of (1.4), the Halley's method (Davies and Dawson, 1975) is obtained in the form:

$$z_i^{(k+1)} = z_i^{(k)} - \frac{p}{p' - \dfrac{pp''}{2p'}} \qquad (2.4)$$

(k = 0, 1,.......)

Setting $\alpha = \frac{1}{n-1}$, a method due to Laguerre (Ostrowski, 1966), and Wilkinson (1965), for instance, can be obtained in the form:

$$z_i^{(k+1)} = z_i^{(k)} - \frac{np}{p' \pm [(n-1)^2 p'^2 - n(n-1)pp'']^{1/2}} \qquad (2.5)$$

(k = 0, 1,.......)

One advantageous point about method (2.4) is, that it is free of the square root sign, which is a disadventage for methods (2.1), (2.3) and (2.5). The presense of square root signs, in these methods, may lead to complex roots even though the roots of the polynomial equation are real, especially when $pp'' > (p')^2$. Furthermore, the cost of evaluating the square root signs may be prohibitively expensive.

The foregoing preliminary discussion was the source of motivation for the present research study.

It was further intended to investigate as to what would happen if α takes on rational values on the interval $(-1, 1)$, i.e., $-1 < \alpha \leq 1$ and the expansion of (1.4) by binomial series. A set of methods was hoped to be obtained out of method (1.4), and as a limiting case of these methods, it was hoped that Halley's formula will be obtained.

$$z_i^{(k+1)} = z_i^{(k)} - \frac{\tfrac{1}{2}p}{(-\tfrac{1}{2}p') \pm [(p')^2 - \tfrac{1}{2}pp'']^{1/2}} \qquad (2.6)$$

First, a plus sign in the term was taken:

$$-\tfrac{1}{2}p' \pm [(p')^2 - \tfrac{1}{2}pp'']^{1/2}$$

The formula (2.6) was then simplified by multiplying through the denominator and numerator parts of the weight function by a factor of 2 to have:

$$z_i^{(k+1)} = z_i^{(k)} - \frac{p}{-p' + [4(p')^2 - 2pp'']^{1/2}} \qquad (2.7)$$

Then the term:

$$[4(p')^2 - 2pp'']^{1/2}$$

was rewritten as:

$$2p'\left[1 - \frac{pp''}{2p'^2}\right]^{1/2}$$

thus, (2.7) becomes:

$$z_i^{(k+1)} = z_i^{(k)} - \frac{p}{-p' + 2p'\left[1 - \dfrac{pp''}{2(p')^2}\right]^{1/2}} \qquad (2.8)$$

(k = 0, 1,.....)

on expanding:

$\left[1 - \frac{pp''}{2(p')^2}\right]^{1/2}$ by the binomial series and writing:

$\frac{pp''}{(p')^2}$ as:

$$\frac{Q}{R}$$

then:

$$\left[1 - \frac{pp''}{2(p')^2}\right]^{1/2} = 1 - \frac{R}{4Q} + \frac{R^2}{32Q^2} + 0(.)^3$$

In view of the substitution $\frac{pp''}{(p')^2} = \frac{R}{Q}$, it is

$$\left[1 - \frac{pp''}{2(p')^2}\right]^{1/2} = 1 - \frac{pp''}{4(p')^2} + \frac{(pp'')^2}{32(p')^4} + 0(.)^3 \quad (2.9)$$

Hence, the method (2.8) after using the right hand side of (2.9), becomes:

$$z_i^{(k+1)} = z_i^{(k)} - \frac{p}{p' - \frac{pp''}{2p'} + \frac{p^2\, pp''^2}{16(p')^3}} \quad (2.10)$$

Similarly, by using the same procedure of (2.10), with α = - 0.1, - 0.2, - 0.3, - 0.4, - 0.6, - 0.7, - 0.8, - 0.9, and 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0, the following iterative methods were obtained.

$$z_i^{(k+1)} = z_i^{(k)} - \frac{p}{p' - \frac{pp''}{2p'} + \frac{9}{80}\frac{p'^2\, pp''^2}{p'^3}} \quad (2.11)$$

(α = - 0.1)

$$z_i^{(k+1)} = z_i^{(k)} - \frac{p}{p' - \frac{pp''}{2p'} + \frac{p'^2\, pp''^2}{10p'^3}} \quad (2.12)$$

(α = - 0.2)

$$z_i^{(k+1)} = z_i^{(k)} - \frac{p}{p' - \frac{pp''}{2p'} + \frac{7}{80}\frac{p'^2\, p''^2}{p'^3}} \quad (2.13)$$

(α = - 0.3)

$$z_i^{(k+1)} = z_i^{(k)} - \frac{p}{p' - \frac{pp''}{2p'} + \frac{3}{40}\frac{p'^2\, p''^2}{p'^3}} \quad (2.14)$$

(α = - 0.4)

$$z_i^{(k+1)} = z_i^{(k)} - \frac{p}{p' - \frac{pp''}{2p'} + \frac{p'^2\, p''^2}{20p'^3}} \quad (2.15)$$

(α = - 0.6)

$$z_i^{(k+1)} = z_i^{(k)} - \frac{p}{p' - \frac{pp''}{2p'} + \frac{3}{80}\frac{p'^2\, p''^2}{p'^3}} \quad (2.16)$$

(α = - 0.7)

$$z_i^{(k+1)} = z_i^{(k)} - \frac{p}{p' - \frac{pp''}{2p'} + \frac{p'^2\, p''^2}{40p'^3}} \quad (2.17)$$

(α = - 0.8)

$$z_i^{(k+1)} = z_i^{(k)} - \frac{p}{p' - \frac{pp''}{2p'} + \frac{p'^2\, p''^2}{80p'^3}} \quad (2.18)$$

(α = - 0.9)

$$z_i^{(k+1)} = z_i^{(k)} - \frac{p}{p' - \frac{pp''}{2p'} + \frac{11}{80}\frac{p'^2\, p''^2}{p'^3}} \quad (2.19)$$

(α = 0.1)

$$z_i^{(k+1)} = z_i^{(k)} - \frac{p}{p' - \frac{pp''}{2p'} + \frac{3}{20}\frac{p'^2\, p''^2}{p'^3}} \quad (2.20)$$

(α = 0.2)

$$z_i^{(k+1)} = z_i^{(k)} - \frac{p}{p' - \frac{pp''}{2p'} + \frac{13}{80}\frac{p'^2\, p''^2}{p'^3}} \quad (2.21)$$

(α = 0.3)

$$z_i^{(k+1)} = z_i^{(k)} - \frac{p}{p' - \frac{pp''}{2p'} + \frac{7}{40}\frac{p'^2\, p''^2}{p'^3}} \quad (2.22)$$

(α = 0.4)

$$z_i^{(k+1)} = z_i^{(k)} - \frac{p}{p' - \frac{pp''}{2p'} + \frac{3}{16}\frac{p'^2\, p''^2}{p'^3}} \quad (2.23)$$

(α = 0.5)

$$z_i^{(k+1)} = z_i^{(k)} - \frac{p}{p' - \frac{pp''}{2p'} + \frac{p'^2\, p''^2}{5p'^3}} \quad (2.24)$$

(α = 0.6)

$$z_i^{(k+1)} = z_i^{(k)} - \frac{p}{p' - \frac{pp''}{2p'} + \frac{17}{80}\frac{p'^2\, p''^2}{p'^3}} \quad (2.25)$$

(α = 0.7)

$$z_i^{(k+1)} = z_i^{(k)} - \frac{p}{p' - \frac{pp''}{2p'} + \frac{9}{40}\frac{p'^2\, p''^2}{p'^3}} \quad (2.26)$$

(α = 0.8)

$$z_i^{(k+1)} = z_i^{(k)} - \frac{p}{p' - \frac{pp''}{2p'} + \frac{19}{80}\frac{p'^2 p''^2}{p'^3}} \quad (2.27)$$

($\alpha = 0.9$)

$$z_i^{(k+1)} = z_i^{(k)} - \frac{p}{p' - \frac{pp''}{2p'} + \frac{p'^2 p''^2}{2p'^3}} \quad (2.28)$$

($\alpha = 1.0$)

On further delection of additional terms of order higher than pp″, in each of these methods (2.10 to 2.28), the Halley's formula was obtained (Hansen and Patrik, 1977).

b. Derivation of Chevbyshev's third order formula. A method using the truncated Taylor polynomial, which followed the Taylor series expansion, was developed for function p given by the relation:

$$0 = p(z_i^{(k)}) + (z_i^{(k+1)} - z_i^{(k)})p'(z_i^{(k)}) + ½(z_i^{(k+1)} - z_i^{(k)})^2 p''(z_i^{(k)}) \quad (3.1)$$

The Halley's correction is:

$$H(z_i^{(k)}) = \frac{p(z_i^{(k)})}{p'(z_i^{(k)}) - \frac{p(z_i^{(k)})p''(z_i^{(k)})}{2p'(z_i^{(k)})}}$$

The following iterative formula is obtained on substituting the term $(z_i^{(k+1)} - z_i^{(k)})^2$ in (3.1) by the Halley's correction:

$$z_i^{(k+1)} = z_i^{(k)} - \frac{p}{p}\left[1 + \frac{2pp'^2 \quad p''}{(2p'^2 - pp'')^2}\right] \quad (3.2)$$

If it is assumed that |p| is sufficiently small, and if the additional term pp in the denominator part of (3.2) is neglected, then (3.2) will result in Chevbyshev's method (Jarratt, 1968):

$$z_i^{(k+1)} = z_i^{(k)} - \frac{p}{p}\left[1 + \frac{pp''}{2p'^2}\right] \quad (3.3)$$

($k = 0, 1, ...$)

It is notewortly that the substitution of $(z_i^{(k+1)} = z_i^{(k)})^2$ by the Chevbyshev's correction in the Taylor polynomial (3.1) was not profitable, as it may be recollected that terms higher than pp″ were ignored in method (3.3). Thus, the optimal method that one can obtain from methods (2.10) – (3.2) is the method (3.3).

c. A new set of methods derivable from Chevbyshev's formula. The presently proposed formulae for finding zeros of non-linear equation of single variable, referred to in the introduction, will now be described in detail.

The term $p''(z_i^{(k)})$, appearing in the Chevbyshev's formula, was approximated in the form:

$$p''(z_i^{(k)}) = \frac{p'(z_i^{(k)}) - p'(z_i^{(k-1)})}{z_i^{(k)} - z_i^{(k-1)}} \quad (4.1)$$

Following carefully such ideas (Braess and Hadeler,1973), it is known that Laguerre's disk:

$$|z^{(k)} - z_i^{(k-1)}| \le n\left|\frac{p(z^{(k)})}{p'(z^{(k)})}\right| \quad (4.2)$$

which contained at least one zero of p. Then using this connection, for optimal z, the inequality (4.2) was satisfied with equality, and the disk was thus in contact with the circle (Braess and Hadeler, 1973):

$$|z_i^{(k)} - z_i^{(k-1)}| = n\left|\frac{p(z^{(k)})}{p'(z^{(k)})}\right| \quad (4.3)$$

In view of the equality expressed in (4.3), method (4.1) may be rewritten in the form:

$$p''(z_i^{(k)}) = \frac{p'(z_i^{(k)}) - p'(z_i^{(k-1)})}{n\frac{p(z^{(k)})}{p'(z^{(k)})}} \quad (4.4)$$

Because of (4.4), the method (3.3) takes the form:

$$z_i^{(k+1)} = z_i^{(k)} - \frac{p(z_i^{(k)})}{p'(z_i^{(k)})}\left[1 + \frac{(p'(z_i^{(k)}) - p'(z_i^{(k-1)})}{2np'(z_i^{(k)})}\right] \quad (4.5)$$

($k = 0, 1,....$)

Since the method (4.5) made use of nonlinear information of the degree of polynomial, there was a unique similarity with Laguerre's method.

Furthermore, an implicit method from method (3.4) can be created.

Suppose, it is instead set as:

$$p''(z_i^{(k)}) = \frac{(p'(z_i^{(k)}) - p'(z_i^{(k+1)})}{(z_i^{(k)}) - (z_i^{(k+1)})} \quad (4.6)$$

then, as before, an implicit iterative formula is obtained:

$$z_i^{(k+1)} = z_i^{(k)} - \frac{p(z_i^{(k)})}{p'(z_i^{(k)})}\left[1 + \frac{p'(z_i^{(k)}) - p'(z_i^{(k+1)})}{2np'(z_i^{(k)})}\right] \quad (4.7)$$

($k = 0, 1,....$)

Method (4.7) is not self-starting. It thus requires the results of other methods. For this purpose, we introduced the use of Newton's second order method that served this purpose as the predictor, while method (4.7) acted as the correction. Hence, in this case, the case of predictor-corrector formula was obtained.

Further information on the use of predictor-corrector formulae may be gained from Lambert (1974), and Kung and Traub (1974). As it were, when the computed $z_i^{(k)}$ from the corrector method was sufficiently close to the result computed from the predictor formula, then it may be noted from the computation that the term:

$$\frac{p'(z_i^{(k)}) - p'(z_i^{(k+1)})}{2np'(z_i^{(k+1)})}$$

was sufficiently close to the origin. In such a circumstance, the method (4.7) differs only very little from Newton's formula:

$$z_i^{(k+1)} = z_i^{(k)} - \frac{p'(z_i^{(k)})}{p'(z_i^{(k)})}$$

(k = 0, 1,....)

and is hence uniformly bounded, away from the origin, on the compact interval containing the zero of p.

d. Numerical experiment. Consider the following numerical test problem.

$p(z) = z^7 - 28z^6 + 322z^5 - 1960z^4 + 6769z^3 - 1312z^2 + 13068z - 5040 = 0$

By taking the initial starting root to $z^{(0)} = 8$

Based on this inital starting root, the values obtained using various methods are shown in Table 1. It may be noted from Table 1 that each of these methods is condensing to the numerical value of 7, which is the true zero of p. The convergence of these methods is monotonic, i.e., $\Phi(z^{(k+1)}) \subseteq \Phi(z^{(k)})$, where $z^{(k+1)} \subseteq z^{(k)}$ as $k \rightarrow \propto$.

Because $\| \Phi(z^{(k+1)})\| \subseteq \| \Phi(z^{(k)})\|$

then $z = \cap \text{ cl } \Phi(z^{(k)})$

where z is the inductive limit of the sequence $(z_i^{(k+1)})$

$\Phi(z^{(k)})$ is bounded below on D_0, $z^{(0)} \subset D_0$

then $\lim_{k\rightarrow\propto} [\Phi(z^{(k)}) - \Phi(z^{(k+1)}] = 0$ on the basis of inverse function theorem.

hence $[\Phi(z^{(k)}) + c(z^{(k+1)}) - z^{(k)})] \leq \Phi(z^{(k)})$, $c \in [0,1]$

thus $z^{(k)} \in c^{(0)}[\Phi(z^{(0)})]$

where it might be defined as $c^{(0)}[\phi(z^{(0)})] = \{z \in D: \phi(z^{(k)}) \leq \phi(z^{(0)})\}$

It is remarked that the orders of convergence of methods (4.5) and (4.7) cannot be less than three in the sense of Alefeld and Herzberger (1974), as well as Kung and Traub (1974). Of all the methods tested, Laguerre's method is the fastest, but one drawback of Laguerre's method is that it requires the computation of square root, which is quite expensive. Amoung the presently proposed methods, method (4.5) appears to be stable, as it does not exhibit the problem of cycling, whereas both Halley's and Chevbyshev's formulae have this tendency to "cycle". Methods (4.5) and (4.7) also do not require second order differentiation of a polynomial equation. As can be observed from the foregoing discussion, the advantages and disadvantages of each of these methods as listed in Table 1 may swing either way.

Table 1. Values for various methods derived from Hansen-Patrik formula (Hansen and Patrik, 1977) for refining zeros of polynomial equation

Number of iterations	Halley's method (2.4)*	Chevbyshev's method**	Proposed method (4.5)*	Proposed method (4.7)*	Newton's method (2.2)*	Laguerre's method (2.5)*
0	8	8	8	8	8	8
1	7.614325063	7.464851443	7.58677686	7.59846325	7.614325069	7.05471012
2	7.162354128	7.137917042	7.335685949	7.302129212	7.568219771	7.000036151
3	7.0009143675	7.011492649	7.151255677	7.1103923	7.291817563	6.999999990
4	7.000001835	7.000015521	7.045419265	7.06378217	7.110719918	7.000000009
5	7.000000529	6.999999646	7.005235768	7.00745791	7.022458237	7.000000019
6		6.999999535	7.00013132	7.000114957	7.001156183	7.000000000
7			6.999999712	7.000000906	7.00000328	
8				7.000000517	6.99999903	

*reference to equation numbers in the text; **Chevbyshev's method (Jarratt, 1968)

Conclusion

The present study investigated the family of iteration formulae for refining zeros of a polynomial presented by Hansen and Patrick (1977), from which the Halley's method was a particular case study. It was noted that when $|p(z)|$ was sufficiently small in magnitude, a family of methods obtained from equation (1.4) for the variable parameter α [$\alpha \in (-1,1)$] was reduced to the well known Halley's method (2.4). By using the Taylor polynomial equation of order two, and substituting the Halley's correction formula, the well known Chevbychev's formula for finding zeros of a polynomial was rediscovered. It is further remarked that the Chevbychev's formula obtained through this process is the optimal method. Furthermore, two new formulae were derived from this Chevbychev's functional iteration method by using Laguerre's disk. Thus, the obtained methods may, and often will, depend on the already calculated values. These methods were illustrated on a polynomial and the results obtained were quite accurate in comparison with other known formulae.

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