Pak J Sci Ind Res 2002 ***45*** *(3) 165-167*

Renewal Process of pH-Distributions with Dual Problem

*Mir G H Talpur, A Fazlani *,Syed A N S Hussani and Raja M Ilyas*

Department of Statistics, University of Sindh, Jamshoro, Pakistan

(Received 21 November 1998; accepted 16 August 2001)

In this paper it is considered to study the two pH-Renewal Process cases by Constructing the vector Markov process, because of the dense of Shi's formula in the set of non-negative random variables, the results are explicit and meaningful for the problems connected with the failure and replacement.

Key words: Markov process, Shi's formula, pH-distribution.

Introduction

Renewal theory was first applied for problems connected with the failure and replacement of equipment. Later however in renewal theory and queuing theory, models developed for one type of application turned out to be useful for applications. Renewal theory deals with the study of renewal process.

A process $\{N(t)\ t \geq 0\}$ whose state space belongs to a denumerable set $\{0,1,2............\}$ and for which the interarrival times $U_n = \sigma'_n - \sigma'_{n-1}$, where n= 1,2....,$\sigma_o = 0$ between successive arrival are positive, is called a renewal process details for discrete case are discussed by Feller (1968), Neuts *et al* (1981) and for continuous random variables (Cox (1955,1962); Feller 1968; Karlin *et al* 1975).

The following two problems are very important in reliability and queueing theory. Let S_n and T_n be renewal times defined by $S_n = \Sigma X_i$ with $S_o = 0$ and

$T_n = \Sigma^n_{i-1} Y_i$ with $T_o = 0$

Assume that $\{X_i\}$ and $\{Y_j\}$ are mutually independent and N is a positive integer.

Problems I. Define R_N = inf $\{t \geq 0/N(t) \geq N\}$, Denote $\theta = N_t(R_N)$ and $\tau = N_2(R_N)$

what is the joint distribution of random variable $\{R.\theta\ \tau\}$?

Problem II. Define ζ_N = INF $\{n/T_n \geq S_N\}$

$$T_o = \sum_{j,I}^{\zeta_N} Y_j \quad N$$

and $\eta_N = N(T_{\zeta N})$

what is the joint distribution of random variables $\{\zeta_N\ T_{\zeta_N}\ \eta_N\}$?

The problem II is called as the dual problem of the problemI. We just study the problem II for case that $\{X_i\}$ and $\{Y_j\}$ are pH-distribution. Before doing this for some special cases the explicit formula of joint distribution have been found.

* Author for correspondence

Shi's formula for transition frequency. The formula obtained by Shi (1994) is about the calculation of transition frequency from one subset to another subset of state space for a continuous time markov chain.

Let S(t) be a continuous Markov chain with state space E={1,2....} and the corresponding infinitesimal matrix $Q=\{q_{ij}\}$ satisfying the condition $\sup_i [q_{ij}] \leq c < \infty$, We use standard notation.

$$P_i(0) = P\{S(O)=i\}$$
$$P_i(t) = P\{S(t)=i\}$$
$$P_{ij}(t) = P\{S(t)=j/S(0)=i\}$$

For absorbing Markov process Shi obtained the following theorem.

Theorem 1.2.1 For the absorbing Markov process S(t), the subset E_0 denotes the absorbing state and $E_1 = E/E_o$ then the p d f of the time absorbing into E_o satisfies the formula.

$$h(t) = H'(t) \sum_{\substack{i \in Ei \\ j \in E_o}} P_i(t)\, q_{ij}$$

Materials and Methods

pH-distribution and pH-renewal process. Erlang first proposed the idea of method supplementary variables. He noted that gamma distributions whose shape parameter is a positive integer may be considered as the probabilities distributions of sum of independent, negative exponential random variables. In this manner very useful results for renewal process of enlarge type can be derived from those of much simpler Poisson process. The memory-less property of the negative exponential distribution is the basic of Erlang. This idea which has been extended by many authors. The notion of Complex-valued probabilities is introduced by Cox (1955), who tries to find phase representation for all probability distribution on the positive real line which have rational Laplace-Stieltjes transforms. Neuts (1982) considered a particular class of probability distribution with rational transforms which are related to finite Markov chains. The emphasis of his discussion is on the algorithmic uses of these probability distributions and involves only classical elementry results from the theory of finite Markov chain, both in discrete and in continuous time.

Discrete case. Discrete pH-distribution are defined by considering an (m + 1) state Markov Chain Transition matrix P of the form (Feller 1968).

$$P=\begin{bmatrix} T & T^{\circ} \\ o & o \end{bmatrix}$$

Where T is a sub stochastic matrix, such that 1-T is nonsingular also $T_{\circ} + T^{\circ}=1$ and the initial probability vector is (α,α_{m+1}) with $(\alpha_{e} + \alpha_{m+1})=1$ where e is the column vector with all its cmponents equal to one, with proper dimension. The probability density $\{P_{K}\}$ of the chain absorbing into the state m+ 1 from any initial state is ki given by

$$P_{\circ}=\alpha_{m+1}$$
$$P_{k} =\alpha\, T^{k-1}\, T^{\circ} \qquad k=1,2,3.........$$

Its probability generating function $P(z)=\alpha_{m+1} +z\alpha\,(i-zT)^{-1}T^{\circ}$ and its factorial moments are given by
$E[X(X+1)+.....(X+k)=k!\,\alpha T^{k-1}\,(I-T)^{-k}\,e$
if k=1 then $E[x]=\alpha\,(I-T)^{-1}e$

Continuous case. We consider a Markov process $\{X(t).\ t \geq o\}$ with infinitesimal generator

$$Q=\begin{bmatrix} T & T^{\circ} \\ o & o \end{bmatrix}$$

here the mxm matrix Tsatisfies $T_{ij} < 0$ for $1 < i < m$, and $T_{ij} > 0$. for x_{j} Also $Te + T^{\circ} = 0$, and the initial probability vector of Q is given y (α,α_{m+1}) with $\alpha_{e} + \alpha_{m+1} =1$ where e is the column vector with all its components equal to one, with proper dimension and the pair (α,T) is called a representation of F (x). We assume that the state 1....m are all transient, so that absorption into the state m+1 from any initial state is certain.

LEMMA.1.1. The state 1...m are transient if and only if the matrix is nonsingular. (Neuts 1981).

Let X is the absorbing time of Markov chain Q, then the distribution of X is

$$F(x) =1-\alpha \exp (T_{X})e, \text{ For } x \geq 0$$

F (x) is called the pH-distribution of order m with representation (α, T).The Laplace-Stieltjes f (s) of F (x) is given by

$$f(s) =\alpha_{m+1} +\alpha\,(sI-T)^{-1}\, T^{\circ} \text{ for } Re(s) \geq 0$$

and the non central moments μi of F (x) are all finite and given by

$$E[X^{i}] = (-1)^{i}\, i!\, (\alpha\, T\text{-}^{i}e)_{.i}$$

Irreducible representation. We consider the version of that Markov process in which the path functions are right-hand continuous, we obtain a Markov process on {1,......,m} with the infinitesimal generator functions are right-hand continuous, we obtain a Mrkov process on {1,......,m} with the

$$^{*}Q=T+T^{\circ}\alpha$$

The representation (α,T) assume $\alpha_{m+1} = 0$ is called irreducible if and only if the matrix q* is irreducible.

Closure properties. (i) If F (x) and G (y) are both continuous (or both discrete) pH-distribution with representations (α,T) and (β,S) of order m and n respectively, then their convolution F.g (.) = H (z) is a pH-distribution with representation (Y, L) of m+n given by (in continuous case)

$$\gamma =[\alpha,\alpha_{m-1}\,\beta]$$

$$L = \begin{bmatrix} \overset{o}{T} & T\beta \\ o & s \end{bmatrix}$$

ii) If F (x) is a pH-distribution with (irreduciable) respresentation (α,T) then

$$F^{*}(x) = 1/\mu' \int_{o}^{X} [1\text{-}F(u)]\, du$$

is a pH-distribution with representation (π,T), where π is the unique solution of the equations $\pi\, O^{*} =0$and $\pi\, e=1$.

pH-Renewal process. The following construction will be very useful. Suppose that upon absorption into the state m+1, we instantaneously perform independent multinomial trials with probabilities $\alpha_{1},.....,\alpha_{m}$, until one of the alternative 1,.....,m occures. Restarting the process Q in the corresponding state, consider the process Q in the corresponding state, consider the time of the next absorption and repeat the same procedure there. It is easy by continuing this procedure indefinitely. We construct a new Markov process in which the state m+1 is an instantaneous state. It is clear that the successive visits to the instantaneous state from a renewal process with underlying distribution F (.), given by F (x) = 1 α exp (Tx)e for $x \geq 0$, a pH-distribution of representation (α, T).

Let $\{N(t), t \geq 0\}$ be the counting process of the pH-renewal process. Denoting the Markov process with generator $Q^{*} = T+T^{\circ}\alpha$ by $\{J(t), T \geq 0)\}$, we introduce the matrices

$P(n,t) = [P_{ij}(n,t)]$,
where $P_{ij}(n,t) = P\{N(t)=n,J+(t)=j\ N(0) =0, J(\phi)=i]$
for $t \geq 0$, $n \geq 0$, $1 \leq i \leq m$, $1 \leq j \leq m$

Neuts (1981) has shown that the matrix generating function

$$P(z,t)= \Sigma_{no}^{\int} P(n,T)z^{n}=\exp[(T+zT^{\circ}\alpha)t]$$

Conclusion

In reliability theory, the distribution of the system first failure time and the number of the failure components before the system failure are related to the renewal process and also the distribution of the busy period and the number of customer served in a busy period in queuing theory. We deal hour with such problems and obtained joint distribution of some random variables in crossing of two renewal processes using vector Markov process. Shi's formula about the calculation of transition frequency from a subset to another subset i.e state space for a continuous time Markov process S (t). The subset E_o denotes the absorbing state and $E_1=E/E_o$ then the p.d. f of the time absorbing/ into E_o and obtained the formula for solving the problem.

F (x) is a pH-distribution with (irreducible) representation (α,T) then $F^*(x)\ 1/\mu_o \int^x [1-F(u)]/du$ is a pH-distribution which provide the unique solution in solving the problem.

It is clear that the pH-renewal process with underlying distribution denoted by the Markov process with the generator $Q^*=T+T^\circ\alpha$ by $\{J(t), t \geq 0\}$ and introduces the matrices which emphasizes, the differences of two renewal process.

References

Beytsm F M 1975 *Probability distributions of phase type, In liber Amicorum Prof. Emeritus* H. Florin, Dept. of Math. Univ. of Loiuvain, Belgium 173-206

Cox D R 1955 A use of complex probabilities in the theory of stochastic processes. *Proc Camb Phil Soc* ***51*** 313-319.

Cox D R 1962 *Renewal Theory,* Methuen London.

Feller W 1968 *An Introduction to Probability Theory and its Applications,* Vol I, 3rd ed, John wiley New York.

Feller W 1971 *An Introduction to Probability Theory and its Application* Vol II, 2nd ed, John Wiley New York

Graham A 1981 *Kronecker Product and Matrix Calcuis with Applications,* John Wiley & Sons.

Karlin S, Taylor H N 1975 *A First Course in Stochastic Processes* 2nd ed. Academic Press, New York.

Keleinroch I 1975 *Queuing Systems,* Vol I, Theory, John Wiley and sons.

Kroese DP 1992 The difference of two renewal processes level crossing and infimum; *Stochastic Models* **8,** 221-243.

Neuts F M 1981 *Matrix-Geometric Solutions In Stochastic Models-An Algorhtimic Approach,* The John Hopkins University Press.

Shi D H 1994 Two counting processes in a Markov chain with continuous time. *Chinese J Appl Prob And Stat* 84-89.

Takacas L 1962 *Introduction to the Theory of Queues,* Oxford University Press, New York.