Pak J Sci Ind Res 2000 ***43*** *(3) 141-144*

THEORY OF THE CAGNIARD-DE HOOP METHOD

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(Received 13 May 1998; accepted May 7, 1999)

The Cagniard-de Hoop method used to simplify certain complex integrals important in engineering and physics has been studied. The path of integration in the complex plane is deformed so that the integral assumes a simple form along the new path. It has been shown that the choice of this path depends on the branch of the multiple valued function as well as the position of the observation point.

Key words: Integrals, Wave propagation, Engineering.

Introduction

Several problems relating to wave propagation in elastic solids are solved by the techniques of the Lapalace or the Fourier transforms. In this approach the main difficulty is encountered in the evaluation of the inversion integral. A typical integral in elastodynamics is of the form

$$u(x,y) = \int_c G(x,y,\xi)\exp[-i\xi x - (\xi^2-k^2)^{1/2}y]d\xi \quad ....(1)$$

where $k > 0$, $y \geq 0$ and C is a contour in the complex ξ-plane (Fung 1965). Integrals similar to (1) occur naturally in problems where an integral representation of Hankel's function $H_n^{(1)}(z)$ or $H_n^{(2)}(z)$ is used (Rawlins 1974; Scheidle *et al* 1978; Mikata 1993). The function $(\xi^2-k^2)^{1/2}$ in Eq.(1.1) is made single valued by introducing branch cuts emanating from the branch points $\pm$ k and going to the left and right along the real axis. In order that the integral may converge, one must choose a branch of the function $(\xi^2-k^2)^{1/2}$ which has a non-negative real part for ξ on C.

$$u(x,y) = \int_c g(x,y,\xi)\exp[-ir\{\xi\cos\theta - i(\xi^2-k^2)^{1/2}\sin\theta\}]d\xi \quad ....(2)$$

where $u(r,\theta) = U(r\cos\theta, r\sin\theta$, and $g(r,\theta;\xi) = G(r\cos\theta, r\sin\theta;\xi)$. It is seldom possible to evaluate the integral (2) exactly and recourse is usually made to some kind of an approximate method. A technique now widely known as the Cagniard-de Hoop method, has been developed to simplify the above integral (Hoop 1960). Recently Mourad and Deschamps (1995) have applied Cagniardc-de Hoop transformation to solve Lamb's problem for a half space having orthorhombic symmetry. The underlying sextic equation in their formalism is similar to the one introduced by Stroh (Ahmed *et al*) and exploited by Barnett and Lothe (1985) to elucidate the existence of a Rayleigh wave in an anisotropic medium. Let

$$S = \xi\cos\theta) - i(\xi^2-k^2)^{1/2}\sin\theta \qquad (3)$$

We are seeking a contour in the ξ-plane whose image under the transformation (3) is a subset of the real axis in the ξ-plane. From eq (3) we get

$$(s - \xi\cos\theta)^2 = -i(\xi^2-k^2)\sin^2\theta$$

which can be rearranged to give

$$(\xi - s\cos\theta)^2 = -(s^2-k^2)\sin^2\theta$$

$$(\xi = s\cos\theta \pm i(s^2-k^2)^{1/2}\sin\theta \quad ....(4)$$

Thus if s is real, $s^2-k^2 \geq 0$ and $\theta \neq 0$, $\pi/2, \pi$, Eq. (4) represents the equation of a hyperbola (Fig 1).

$$\left[\frac{\mathrm{Re}\xi}{k\cos\theta}\right]^2 - \left[\frac{\mathrm{Im}\xi}{k\sin\theta}\right]^2 = 1, \quad ....(5)$$

i.e. a hyperbola with its centre at the origin and its foci at the branch points of the function $(\xi^2-k^2)^{1/2}$. The path which we are seeking to simplify the integral (2) is a branch of this hyperbola. For further reference, let us denote this hyperbola by H and its respective branches which lie in the left and right half planes by H_L and H_R.

Here we shall investigate the transformation (3) which lies at the root of the Cagniard-de Hoop method. We shall show that the choice of a branch of the hyperbola H is not arbitrary. It depends on the contour C as well as the location of the observation point (x,y). We find that the tgransformation (3) transforms either H_L or H_R but not both into a subset of the real s-axis. If the image of H_L (or H_R) is a ray, the image of H_R (or H_L) is a subset of another hyperbola. Since the success of

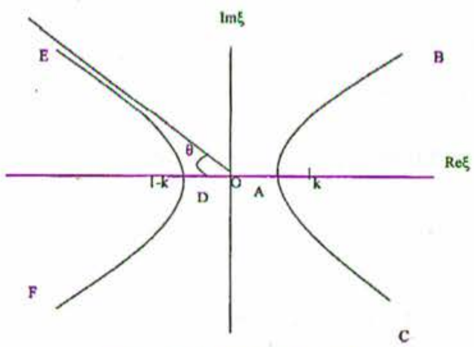

Fig 1. The hyperbola H

the method hinges on the result of the transformation being real, the decision whether to choose H_L or H_R must be made with caution.

1. Branches of the function $(\xi 2\text{-}k^2)^{1/2}$

We briefly discuss the mapping properties of the two branches of the function $(\xi^2\text{-}k^2)^{1/2}$.

Let $r_1 = |\xi - k|$, $r_2 = |\xi + k|$, $\theta_1 = \arg(\xi\text{-}k)$, $\theta_2 = \arg(\xi\text{-}k)$, then a branch $f_1(\xi)$ of the function $(\xi^2\text{-}k^2)^{1/2}$ can be chosen as (Fig 2; Chunchill 1995)

$$f_1(\xi) = \sqrt{r_1 r_2}\exp\left\{\frac{i(\theta_1\text{-}\theta_2)}{2}\right\} \quad \text{.......(1.1)}$$

where

$$r_1 > 0, r_2 > 0, 0 < \theta_1 < 2\pi \text{ and } -\pi < \theta 2 < \pi.$$

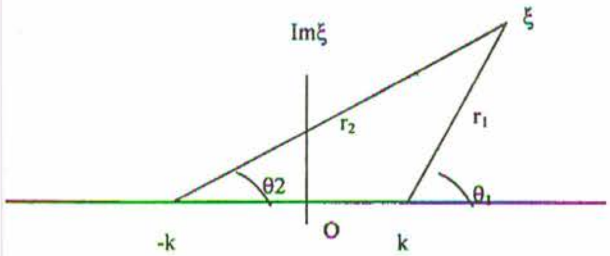

Fig 2. Choice of the branch of $(r_2 - k_1)^{1/2}$

The second branch $f_2(\xi)$ of the function $(\xi^2\text{-}k^2)^{1/2}$ can be chosen as

$$f_2(\xi) = \sqrt{r_1 r_2}\exp\left\{\frac{i(\theta_1\text{-}\theta_2)}{2}\right\} \quad \text{.......(1.2)}$$

where $r_1 > 0, r_2 > 0, 2\pi < \theta_1 < 4\pi$ and $-\pi < \theta_2 < \pi$, It is clear that $f_2(\xi)) = -f_1(\xi)$ for every ξ not lying on the branch cut

Now consider the mapping of the ξ-plane by the function f_1. If ξ_1 is a point in the first quadrant then $0 < \theta_1 + \theta_2 < \pi$ and Eq (1.2) shows that $0 < \arg f_1(\xi_1) < \pi/2$. Thus f_1 maps the first quadrant into itself. Similarly we can show that the function maps the second quadrant into itself, and the third and the fourth quadrants into the first and second quadrants respectively. It maps each of the line segments $-k < \mathrm{Re}(\xi) \le 0$ and $0 \le \mathrm{Res}(\xi) < k$ onto the segment $0 < \mathrm{Im}(\xi) \le k$. Also it maps each of the + ve and -ve imaginary axes onto the ray $\mathrm{Im}(\xi) > k$.

Since $f_2(\xi) = -f_1(\xi)$, it follows that $f_2(\xi)$ maps the first quadrant into the third, the second quadrant into the fourth and each of the third and fourth quadrants into itself. The line segments $-k < \mathrm{Re}(\xi) \le 0$ and $0 \le \mathrm{Re}(\xi) < K$ are both mapped onto the segment $-k \le \mathrm{Im}(\xi)\ 0$. The function maps each of the +ve and -ve imaginary axes onto the ray $\mathrm{Im}(s) < -k$.

2. Image of the hyperbola

Let us assume $0 < \theta < \pi/2$, so that both $\sin\theta$ and $\cos\theta$ are positive. Also let

$$f_1(\xi) = (\xi^2\text{-}k^2)^{1/2} \quad \text{.......(2.1)}$$

where f_1 is the same as in Sec.1. Consider the image of the hyperbola (5) under the transformation (3), i.e

$$s = \xi\cos\theta - if_1(\xi)\sin\theta \quad \text{.......(2.2)}$$

Any point on the arc AB can be represented by

$$\xi_1 = a\cos\theta + i\sqrt{a^2\text{-}k^2}\sin\theta \quad \text{.......(2.3)}$$

where $a \le k$. Since

$$\xi^2_1 - k^2 = [\sqrt{a^2\text{-}k^2}\cos\theta + ia\sin\theta]^2 \quad \text{.......(2.4)}$$

we find

$$f_1(\xi_1) = \sqrt{a^2\text{-}k^2}\cos\theta + ia\sin\theta]^2 \quad \text{.......(2.5)}$$

where the value $f_1(\xi_1)$ has been chosen in accordance with the mapping properties of the function f_1, discussed in Sec.1. If we denote the image of ξ_1 under (2.2) by s_1, we find from (2.2), (2.3) and (2.5)

$$s_1 = a. \quad \text{.......(2.6)}$$

Since $a \ge k$ is arbitrary, we see that the arc AB of the hyperbola of Fig. 1 is mapped one-to-one onto the ray $\mathrm{Re}(s) \ge k$.

Now consider a point ξ_4 on the arc AC of the hyperbola and denote its image by s_4. We may take.

$$\xi_4 = a\cos\theta - i\sqrt{a^2\text{-}k^2}\sin\theta$$

where $a \ge k$. Now

$$f_1(\xi_4) = -\sqrt{a^2\text{-}k^2}\cos\theta + ia\sin\theta$$

and

$$s_4 = \xi_4\cos\theta - if_1(\xi_4)\sin\theta = a.$$

We again find that arc AC is also mapped one-to-one onto the ray $\mathrm{Re}(s) \ge k$ in the s-plane.

Now consider the arc DE of the hyperbola (2.5) (Fig. 1). Let

$$\xi_2 = -\cos\theta + i\sqrt{a^2\text{-}k^2}\sin\theta \quad \text{.......(2.7)}$$

be a point on DE, with $a \ge k$. Proceeding as in (2.4) and (2.5) we find, in this case,

$$f_1(\xi_2) = -\sqrt{a^2\text{-}k^2}\cos\theta + ia\sin\theta \quad \text{.......(2.8)}$$

From (2.2), (2.7) and (2.8) we get

$$s_2 = \xi_2\cos\theta - if_1(\xi_2)\sin\theta = -a\cos 2\theta + i\sqrt{a^2 - k^2}\sin 2\theta \quad \text{..(2.9)}$$

It is apparent from the above equation that the image of the arc DE is not a ray in general, instead it is mapped onto (i) the arc of the hyperbola,

$$\left[\frac{\mathrm{Re}(s)}{k\cos 2\theta}\right]^2 - \left[\frac{\mathrm{Im}(s)}{k\sin 2\theta}\right]^2 = 1 \quad \text{.......(2.10)}$$

which lies in the second quadrant, if $\theta \geq \pi/4$,

(ii) the positive imaginary axis $Im(s) \geq 0$ if $\theta = \pi/4$.,

(iii) the arc of the hyperbola (2.10) which lies in the first quadrant, if $\pi/4 < \theta < \pi/2$.

The image of the arc DF of Fig. 1 is found, in a similar manner to be subset of the hyperbola (2.10).

The purpose of the Cagniard-de Hoop method is to simplify the integral (2). If the branch of $(\xi^2\text{-}k^2)^{1/2}$ is chosen as the function $f_1(\xi)$, this object cannot be achieved by deforming the path of integration C to the arc H_L of the hyperbola of Fig. 1. It can be easily shown that the mapping (2.2) does not map any point of the left half ξ-plane in its domain into the real s-axis.

Instead of (2.2), let us now consider the mapping

$$s = \xi \cos\theta - if_2(\xi)\sin\theta \quad \text{...................................(2.11)}$$

Using analogous arguments, we can establish kthat the mapping (2.11) maps each of the arcs DE and DF (Fig.1) onto the ray $Re(S) \leq -k$ while the branch H_R is mapped into the hyperbola (2.10).

From the above discussion, it is clear that in an application of the Cagniard-de Hoop method, the choice $f_1(\xi)$ requires path of integration to be deformed into the branch H_R while the choice $f_2(\xi)$ necessitates the selection of the branch H_L of the hyperbola of Fig.1.

Thus far we have assumed $0 < \theta < \pi$. Now let $\pi/2 < \theta < \pi$, so that $\cos\theta < 0$ and $\sin\theta > 0$. In this case

$$\xi_2 = a\cos\theta + i\sqrt{a^2 - k^2}\,\sin\theta$$

with $a \geq k$, is a point on H_L. We find

$$f_1(\xi_2) = \sqrt{a^2 - k^2}\cos\theta + ia\sin\theta$$

and

$$s_2 = \xi_2\cos\theta - if_1(\xi_2)\sin\theta = a$$

Thus we see that for $\pi/2 < \theta < \pi$ the mapping (2.2) maps the arc DE of the hyperbola (5), (Fig.1), one-to-one onto the ray $Re(s) \geq k$. We can also show that the arc DF is also mapped onto the same ray. On the other hand, the branch H_R is mapped into hyperbola (2.10) by (2.2).

For the Cagniard-de Hoop method to be successful the appropriate choice of the branch of the hyperbola (5), which should replace the contour C in the integral (2), has to be made as shown in Table 1.

Table 1

Choice of H_L or H_R

Value of θ	Branch of $(\xi^2\text{-}k^2)^{1/2}$	Branch of the hyperbola
$0 < \theta < \frac{\pi}{2}$	F_1	H_R
$0 < \theta < \frac{\pi}{2}$	F_2	H_L
$\frac{\pi}{2}\ 0 < \theta < \pi$	F_1	H_L
$\frac{\pi}{2}\ 0 < \theta < \pi$	f_2	H_R

3. An example of the Cagniard-de Hoop method

Let us consider a simple example of the integral (2) by taking The above integral arises in the problem of a half-space

$$g(r, \theta; \xi) = -\frac{1}{2\pi}\frac{1}{(\xi^2 - k^2)^{1/2}} \text{ , thus getting}$$

$$u(r,\theta) = -\frac{1}{2\pi}\int_C \left\{ \frac{\text{Exp}[-ir\{\xi\cos\theta - i(\xi^2 - k^2)^{1/2}\sin\theta\}]}{(\xi^2 - k^2)^{1/2}} \right\} d\xi \quad (3.1)$$

subjected to antiplane surface disturbances. The contour C cannot be entirely above the real axis or below it because none of $f_1(\xi)$ and $f_2(\xi)$ would then have a positive real part along C. We must choose it as one of the two contours shown in Fig.3.

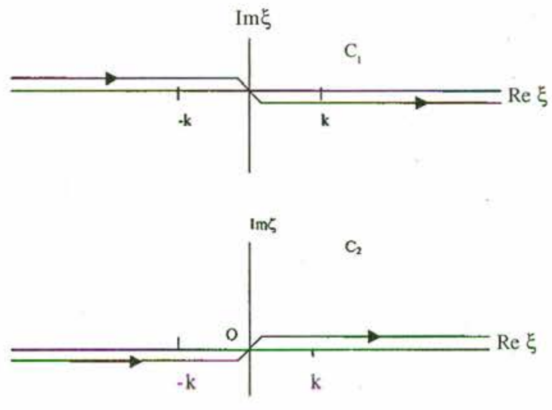

Fig.3 Choice of a contour.

If the contour C_1 is chosen, then the branch of $(\xi^2\text{-}k^2)^{1/2}$ which has positive real part along C_1 is f_2. The integral becomes

$$u(r,\theta) = -\frac{1}{2\pi}\int_{C_1} \left\{ \frac{\text{Exp}[-ir\{\xi\cos\theta - if_2(\xi)\sin\theta\}]}{f_2(\xi)} \right\} d\xi$$

From Table 1 we find that, for $0 < \theta < \pi/2$, we can deform the contour C_1 into the branch H_L of the hyperbola and, for $\pi/2 < \theta < \pi$, we should choose H_R for this purpose. Let $0 < \theta < \pi$. The contour can be deformed by drawing arcs of large radii, since the integral along these arcs vanishes when these radii approach infinity (Fig.4).

No singular point of the integrand is encircled and the integral (3.2) is the same, except for a change of sign, when C_1 is re-

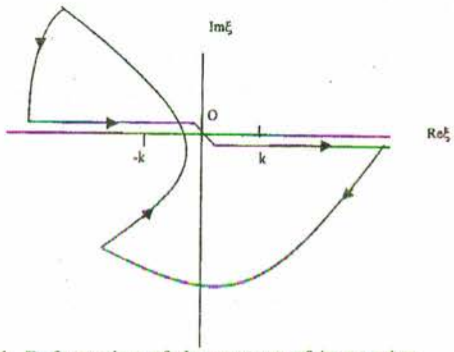

Fig.4 Deformation of the contour of integration.

placed by the arc FDE of the hyperbola (5). Now for $-\infty < s \leq -k$, on DE we have

$$\xi = s\cos\theta + i\sqrt{s^2 - k^2}\,\sin\theta \quad \text{.................(3.3)}$$

$$f_2(\xi) = \sqrt{a^2 - k^2}\,\cos\theta + is\,\sin\theta$$
$$\xi\,\cos\theta - i f_2(\xi)\sin\theta = s$$

And on FD

$$\xi = \cos\theta - i\sqrt{s^2 - k^2}\,\sin\theta \quad \text{.................(3.4)}$$
$$f_2(\xi) = -\sqrt{s^2 - k^2}\,\cos\theta + is\sin\theta$$
$$\xi\,\cos\theta - i f_2(\xi)\sin\theta = s$$

Now the integral (3.2) becomes

$$u(r,\theta) = -\frac{1}{2\pi}\int_{FD+DE}\left\{\frac{\mathrm{Exp}[-ir\{\xi\cos\theta - if_2(\xi)\sin\theta\}]}{f_2(\xi)}\right\}d\xi$$

$$= \frac{1}{2\pi}\int_{-\infty}^{-k}\left\{\frac{\mathrm{Exp}\,(-irs)}{-\sqrt{s^2-k^2}\cos\theta + \sin\theta}\right\}\frac{d\xi}{ds}\,ds$$

$$+\ \frac{1}{2\pi}\int_{-k}^{-\infty}\left\{\frac{\mathrm{Exp}\,(-irs)}{\sqrt{s^2-k^2}\cos\theta + is\,\mathrm{in}\theta}\right\}\frac{d\xi}{ds}\,ds \quad \text{.........(3.5)}$$

In the first integral on the right side of (3.5)

$$\frac{d\xi}{ds} = \frac{\sqrt{s^2-k^2}\cos\theta + is\sin\theta}{\sqrt{s^2-k^2}}$$

While in the second

$$\frac{d\xi}{ds} = \frac{\sqrt{s^2-k^2}\cos\theta + is\sin\theta}{\sqrt{s^2-k^2}}$$

Thus (3.5) becomes

$$u(r,\theta) = -\frac{1}{\pi}\int_k^{\infty}\left\{\frac{\mathrm{Exp}\,(iru)}{\sqrt{u^2-k^2}}\right\}du$$

$$= \frac{i}{\pi}H_o^{(1)}(Kr)$$

where $H_o^{(1)}$ (kr) denotes the Hankel function of first kind.

Now consider the choice of C_2 (Fig.3) as the contour of integration in (3.1). We must choose the branch f_1 of the multiple valued function $(\xi^2 - k^2)^{1/2}$ since it has positive real part on this contour. The integral

$$u(r,\theta) = -\frac{1}{2\pi}\int_{C_2}\left\{\frac{\mathrm{Exp}[-ir\{\xi\cos\theta - if_1(\xi)\sin\theta\}]}{f_1(\xi)}\right\}d\xi$$

is evaluated using analogous steps. However the contour is now deformed into the arc HR of the hyperbola. Omitting details, we finally end up with

$$u(r,\theta) = \frac{i}{2\pi}\ H_o^{(2)}(kr)$$

where $H_o^{(2)}$ (kr) is the Hankel function of the second kind.

The above calculations show on the one hand that, if care is not exercised in the choice of the proper branch of the function $(\xi^2-k^2)^{1/2}$ or the branch H_L or H_R of the hyperbola one may obtain an erroneous result.On the other hand the flexibility in the choice of the branches is useful, because the physics of a problem may force us to choose a solution with proper asymptotic behaviour. If the time dependence is harmonic of the form exp(-iωt) the solution (4.6) is suitable for an outgoing cylindrical wave while for an incoming wave the solution (4.7) will be appropriate.

References

Barnett D M, Lothe J 1985 "Free surface (Rayleigh) waves in anisotropic half-spaces: The surface imnpedence method, *Proc R Soc London* **A402** 135-152.

Churchill R V, Brown J W, Verhey R F 1995 *Complex Variables and Applications,* McGraw-Hill, London.

De Hoop A T 1960 "A modification of Cagniard's method for solving seismic pulse problems" *Appl Sco Res* **B8** 349-356.

Fung Y C 1965 *Foundations of Solid Mechanics*, Prentice Hall.

Mikata Y 1993 "Reflection and Transmission by a periodic array of coplanar cracks: normal and oblique incidence", *J Appl Mech* **60** 911-919.

Miklowitz J 1978 *Elastic Waves and Wave Guides,* North Holland, Amsterdam.

Mourad A, Deschamps M 1995 "Study of 3D Lamb problem for an anisotropic hal: space by the Cagniard-de Hoop methods", *J Acoust Soc Am* **97** 3194-3197.

Rawlins A D 1974 "Acoustic diffraction by an absorbing semi-infinite half plane in a moving fluid", *Proc R Sco Edinb* **A72** 337-357.

Scheidle W , Ziegler F 1978 "Interaction of a pulsed Rayleigh surface wave and a rigid cylindrical inclusion", in *Modern problems in Elastic Wave Propagation* eds J Miklowitz and J D Achenbach. John Wiley and Sons, New York pp 145-169.

Stroh A N 1962 "Steady state problems in anisotropic elasticity" *J Math Phys* **41** 77-103.