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On Condensing Mappings in D-Metric Spaces

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In this paper two measures of noncompactness in D-metric spaces are defined and some fixed point theorems are proved for condensing mappings in D-metric spaces.

Key words. Condensing mapping, D-metric, Fixed point.

Introduction

Recently the present author (Dhage 1992) has introduced the notion of D-metric spaces as follows. Let x be a non-empty set. A real function D on x × x × x is said to be a D-metric on x if it satisfies the following properties.

(M_1) $D(x, y, z) \geq 0$ for all x, y, z ε x and equality holds if and only if $x = y = z$.

(M_2) $D(x, y, z) = D(y, x, z) = ...$; (symmetry)

(M_3) $D(x, y, z) = D(x, y, a) + D(x, a, z) + D(a, y, z)$, for x, y, z, a ε x. (rectangle inequality)

A non-empty set x together with a D-metric, D is called the D-metrics space and it is denoted by (x, D). The generalization of the D-metric as a function of n variables is given in (Dhage 1984 and 1992). A sequence x_n in the D-metric space x is called D-cauchy if $\lim D(x_m, x_n, x_p)= 0$, A sequence $\{x_n\}$ in a D-metric space x is said m, n, p →∝.
to be D-convergent and converges to a point x in x if $\lim D(x_m, x_n, x) = 0$. A complete m, n →∝

D-metric space x is one in which every D-cauchy sequence in x converges to a point in x. Let x_o ε x and let $\varepsilon \geq 0$ be given. Than a ball $B(x_o, \varepsilon)$ centered at x_o of radius e in x is defined by

$B(x_o, \varepsilon) = \{ y, \varepsilon\ x \mid D(x_o, y, y) < \varepsilon$ and if y, z ε $B(x_o, \varepsilon)$ are any two into then $D(x_o y,z) \geq \varepsilon\}$. (1.1)

Then the collection $\{B(x, \varepsilon): x\ \varepsilon\ x\}$ of all ε - balls induces the topology τ on x called the D-metric topology on x provided D satisfies the condition

(M_4) $D(x, z, z) \geq D(x, y, y) + D(y, z, z)$
for all x, y, z ε x.

The topology τ is same as the topology of D-metric convergence in x. The topological propcrties of a D-metric space x are similar to a ordinary metric space and the details are given in (Dhag 1994). The collection $\{B(x, \varepsilon) : x\ \varepsilon\ x\}$ forms the open cover for the sat x. If this open cover has a finite subcover, i.e., if there exist finite points $x_1, x_2 ...., x_n$ in x such that $X C U_{i=1}^{n} B(x_1, \varepsilon)$, the x is called the compact D-metric space. If x is a compact D-metric space, then every sequence $\{x_n\}$ in x has a convergent subsequence. Let A be non-empty set in x. Than the diameter of A denoted by diam (A) is defined by

$$\text{diam}(A) = \sup \{D(a, b, c) : a, b, c\ \varepsilon\ A\} \quad .... (1.2)$$

A subset A of the D-metric space x is said to be bounded if there is diam (A) ≤ M. Since the compact spaces have some nice properties and are easy to deal with, several results are possible in compact D-metric spaces. Therefore, it is of interest to measure the noncompactness of non-empty and bounded sets in a D-metric space x. Below we state two measures or noncompactness of a bounded set in the D-metric spaces on the lines of kuratowskii (1930) and (Petrysyn 1971) measures of noncompactness in the ordinary metric spaces.

Definition 1.1: The set measure of noncompactness of a bounded set A in a D-metric space x is a nonnegative real number α(A) defined by

$$\alpha(A) = \text{in}\varepsilon \{r > 0: A = U_{i=1}^{n} A_i, \text{diem}(A_1) \leq r, \text{vi}\} ......(1.2)$$

Definition 1.2: The ball measure of noncompactness of a bounded set A in a D-metric space x is a nonnegative real number β (A) defined by

$$\beta(A) = \inf\{ r > 0 : AC\ U_{i=1}^{n} B(x_i, r), x_i\ \varepsilon\ x\} \quad .... (1.3)$$

The measures of noncompactness α and β have similar properties.

Below we state some properties of the measure of noncompactness.

Lemma 1.1: For any subsets A and B of x.

(i) α (A) = 0 if and only if A is compact.

(ii) $A \subset B => \alpha(A) \leq \alpha(B)$

(iii) $\alpha(A \cup B) = \max\{\alpha(A), \alpha(B)\}$
(iv) $\alpha(A \cap B) = \min\{\alpha(A), \alpha(B)\}$
(v) $\alpha(\bar{A}) = \alpha(A)$, where $\bar{A}$ is the closure of A.

Proof: The proof is similar to the properties of kuratoiskii measure of noncompactness in ordinary metric spaces. we omit the details.

Remark. 1.1 we note that in the special case when the D-metric, D is defined on a non-empty set x by

$$D(x, y, z) = \max\{d(x, y), d(y, z), d(z, x)\} \quad (1.5)$$

where d is a ordinary metric on x, then the diameter of a bounded set A in the D-metric space x is just reduced to the diameter of in the ordinary metric space (x, d) given by

$$\text{diam}(A) = \max\{d(x, y) : x, y \in A\} \quad (1.6)$$

In this case the set and ball measures of noncompactness in the D-meteric space x are reduced respectively to the kuratowskii measures of noncompactness in the ordinary metric space x.

In the following section we prove the main results of this paper. In the sequel, by x we always mean, unless otherwise specified, the D-metric space with D-metric D.

Results and Discussion

Definition 2.1: A mapping T, $x \to x$ is called k-set contraction if for any bounded set A in x and $\alpha(TA) \le k\,\alpha(A)$ for some $k > 0$.

Definition 2.2. A mapping $T : x \to x$, is said strict-set contraction if it is a k-set contraction with $k < 1$.

Definition 2.3: A mapping $T : x \to x$ is said to be condensing if for any bounded set A in x and $\alpha(TA) < \alpha(A)$, $\alpha(A) > 0$.

Theorem 2.1 lst $T : x \to x$, x_a complete and bounded D-metric space, be a continuous and condensing mapping. Then T has a fixed point.

Proof: The proof is similar to a theorem of Fury and Vignoli (1989). In ordinary metric spaces. we omit the details.

As a consequence of theorem 2.1, we obtain the following corollaries:

Corollary 2.1. Let $T: x \to x$, $x_a$complete bounded D-metric space, be a continuous and strict - set contraction mapping. Then T has a fixed point.

Corollary 2.2. (Fury and Vignoli 1989) : Let $T : x \to x$, x_a complete bounded ordinary metric space, be a continuous and condensing mapping. Then T has a fixed point.

Proof: Define a D-metric D on x by (1.5). Then by Remark 1.1. a mapping T which is condensing in ordinary metric space x, is also condensing in the D-metric space x. Again, the continuity of T in ordinary metric space implies the continuity of T in the D-metric spaces. Now the conclusion follows by an application of Theorem 2.1.

Theorem 2.2: Let $T : x \to x$, x_a complete bounded D-metric space, be a mapping satisfying

$$D(Tx, Ty, Tz) \le \phi(D(x, y, z)) \quad (2.1)$$

for all x, y, z $\in$ x, where ϕ is a continuous real function such that $\phi(r) < r$, $r > 0$. Then T has aunique fixed point.

Proof: By theorem 2.1 of (Dhage 1994), T is continuous on x . We show that T is condensing on x. Let A be a bounded subset of x. Let $\varepsilon > 0$ be given and suppose $A = \cup_{i=1}^{n} A_i$

Then $T(A) = \cup_{i=1}^{n} T(A_i)$. Now definition of α, we get diam $(A_i) < \alpha(A) + \varepsilon$, for all i, i = 1, 2, ..., n. By inequality (2.1), we obtain

$$\begin{aligned}\alpha(TA) &\le \text{diam}(TA_i)\\ &\le \phi(\text{diam}(A_i))\\ &\le \max\{\phi(t) : t \in [\alpha(A), \alpha(A) + \varepsilon]\}\\ &= \phi(\alpha(A))\\ &< \alpha(A), \alpha(A) > 0.\end{aligned}$$

This shows that T is a condensing on x. Now an application of theorem 2.1. Yields that T has a fixed point. The uniqueness of fixed point follows from the condition (2.1). This completes the proof.

Corollary 2.4: Let $T : x \to x$, x a complete bounded D-metric space, be a mapping such that there exists a $p \in N$ satisfying

$$D(T^p x, T^p y, T^p z) \le \phi(D(x, y, z) \quad (2.2)$$

for all x, y, z $\in$ x, where ϕ is a continous real function such that $\phi(r) < r$, $r > 0$. Then T has a unique fixed point.

If $\phi(r) = kr$, $0 \le k < 1$, then theorem 2.2 includes the following result as a corollary proved by present author (Dhage 1992) with a different method.

Corollary 2.5: Let $T : x \to x$, x a complete bounded D-metric space, be a mapping satisfying

$$D(Tx, Ty, Tz) \le k\,D(x, y, z) \quad (2.3)$$

for all x, y, z $\in$ x and $0 \le k < 1$. Then T has a unique fixed point.

Open problem: It is an open problem whether theorem 2.2. can be proved by other method without using the measure of noncompactness.

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