Pak J Sci Ind Res 1999 **42** *(5) 242-243*

COMMON FIXED POINT THEOREMS IN 2-METRIC SPACE

Sushil Sharma

Department of Mathematics, Govt Kalidas Girls College, Ujjain-456001, India

(Received 25 August 1994; accepted 20 June 1995)

The object of this paper is to prove a fixed point theorem of Singh and Meade type (1977) in 2-metric space.

Key words: Fixed point theorems, Housdroffmetric, 2-Metric space.

Introduction

Gahler (1963) has introduced the motion of a 2-metric space as follows:

A 2-metric space is a non empty set X together with a real valued function d on X x X x X satisfies the following properties:

(i) For two distinct points x, y in X, there exists a point z in X such that $d(x, y, z) \neq 0$.

(ii) $d(x, y, z) = 0$ if at least two of x, y, z are equal.

(iii) $d(x, y, z) = d(x, z, y) = d(y, z, x)$

(iv) $d(x, y, z) \leq d(x, y, u) + d(x, u, z) + d(u, y, z)$ for all x, y, z, u in X.

Definition 1.1: sequence $\{x_n\}$ in a 2-metric space (X, d) is said to be Cauchy sequence if $\lim_{m,n\to\infty} d(x_m, x_n, z) = 0$ for all z in X.

Definition 1.2. A 2-metric space (X, d) is said to be complete if every Cauchy sequence in X is convergent.

Lemma 1.1. (Cho *et al* 1988) For every i, j, k ∈ N, $d(x_i, x_j, x_k) = 0$ where $\{x_n\}$ is the sequence in X defined in the proof of the theorem.

Results and Discussion

Throughout this paper (X, d) stands for a complete 2-metric space. Further, ϕ is the set of functions $\phi . (R^+)^5 \to R^+$, which are upper semi-continuous from the right and non-decreasing in each coordinate variable such that $\phi(t, t, t, at, bt) < t$ for each $t > 0$ and $a \geq 0$, $b \geq 0$ with $a + b \leq 2$.

In this note we extend result of Singh and Meade (1977) for 2-metric space. We prove the following:

Theorem 1. Let (X, d) be a complete 2-metric space and let S and T be self mapping of X. Suppose there exists a $\phi \in \phi$ such that for all x, y, a ∈ X, $d(Sx, Ty, a) \leq \phi[d(x, y, a), d(x, Sx, a), d(y, Ty, a), d(x, Ty, a), d(y, Sx, a)]$....(2.1) holds, then there exists a z ∈ X such that z is a unique common fixed point of S and T.

Proof. Let x ∈ X be any point. Then define a sequence of iterates $\{x_n\}$. First choose a point x_1 in $X_1 = S_x$. Then choose a point x_2 in $X_2 = Tx_1$ and so on. In general choose a point x_{2n-1} in $X_{2n-1} = Sx_{2n-2}$ and a point x_{2n} in $X_{2n} = Tx_{2n-1}$ for n = 1,2,3,...... Put $V_n = d(X_n, X_{n+1}, a)$. We shall prove that $\{V_n\}$ is a decreasing sequence.

If $V_{2n+1} > V_{2n}$ then by (2.1) we have

$$V_{2n+1} = d(X_{2n+1}, X_{2n+2}, a) = d(Sx_{2n}, Tx_{2n+1}, a).$$
$$\leq \phi\, [d(x_{2n}, x_{2n+1}, a), d(x_{2n}, Sx_{2n}, a), d(x_{2n+1}, Tx_{2n+1}, a), d(x_{2n}, Tx_{2n+1}, a), d(x_{2n+1}, Sx_{2n}, a)]$$
$$\leq \phi\, (V_{2n}, V_{2n}, V_{2n+1}, V_{2n} + V_{2n+1}, 0).$$
$$< V_{2n+1} \text{ as } V_{2n+1} > 2n.$$

giving a contradiction.

Hence $V_{2n+1} < V_{2n}$. Similarly we can show that $V_{2n+2} < V_{2n+1}$ Thus $\{V_n\}$ is a decreasing sequence.

Now since

$$V_1 = d(X_1, X_2, a) = d(Sx, Tx_1, a)$$

$$\leq (V_0, V_0, V_0, 2V_0, 0).$$

it follows by induction that

$$V_n \leq \psi^n (V_0), \text{ where}$$

$$\psi(t) = \max\, \{\phi(t, t, t, 2t, 0), \phi(t, t, t, 0, 2t)\}$$

Then by a lemma of Matkowsk (1977) we have

$$\lim_{n\to\infty} \quad V_n = 0 \text{.......................(2.2)}$$

We now show that $\{x_n\}$ is a Cauchy sequence. For this, it suffices to prove that $\{x_{2n}\}$ is a Cauchy sequence.

Suppose that $\{x_{2n}\}$ is not a Cauchy sequence. Then there is an $\in > 0$ such that for each even integer 2k, there are even integers $2m(k) > 2n(k) > 2k$ such that

$$d(x_{2m(k)}, x_{2n(k)}, a) > \in \text{.......................}(2.3)$$

By the well ordering principle, for each even integer 2k, let 2m(k) be the least even integer exceeding 2n(k) satisfying (2.3). That is to say that

$$d(x_{2n(k)}, x_{2m(k)-2}\, a) \le \in \text{ and (2.3) holds......}(2.4)$$

Now

$$\in < d(x_{2n(k)}, x_{2m(k)}, a) \le d(x_{2n(k)}, x_{2m(k)-2}, a) + V_{2m(k)-2} + V_{2n(k)-1}$$
(using lemma 1.1)

Then by (2.3) and (2.4) it follows that

$$\lim_{k\to\infty} d(x_{2n(k)}, x_{2m(k)}, a) = \in \text{ ....}(2.5)$$

Also by the triangle inequality and using lemma 1.1 we have

$$|d(x_{2n(k)}, x_{2m(k)-1}, a) - d(x_{2n(k)}, x_{2m(k)}, a)| \le V_{2m(k)-1}$$

and

$$|d(x_{2n(k)+1}, x_{2m(k)-1}, a) - d(x_{2n(k)}, x_{2m(k)}, a)| \le V_{2m(k)-1} + V_{2n(k)}$$

Using (2.5) we get

$$d(x_{2n(k)}, x_{2m(k)-1}, a) \to \in$$

and

$$d(x_{2n(k)+1}, x_{2m(k)-1}, a) \to \in$$

But by the hypothesis (2.1) one gets

$$d(x_{2n(k)}, x_{2m(k)}, a) \le d(_{2n(k)}, x_{2n(k)+1}, a) + d(x_{2n(k)+1}, x_{2m(k)}, a) + d(x_{2n(k)+1}, x_{2m(k)}, a) + d(x_{2n(k)}, x_{2m}(k), X_{2n(k)+1})$$

The last term will be zero by lemma 1.1

$$= d(x_{2n(k)}, x_{2n(k)+1}, a) + d(x_{2n(k)+1}, x_{2m(k)}, a)$$
$$\le V_{2n(k)} + d(Sx_{2n(k)}, Tx_{2m(k)-1}, a),$$
$$\le V_{2n(k)} + \phi[d(x_{2n(k)}, x_{2m(k)-1}, a)\, d(x_{2n(k)}\, Sx_{2n(k)}, a)\, d(x_{2m(k)-1}, Tx_{2m(k)-1}, a), d(x_{2n(k)}, Tx_{2m(k)-1}, a), d(x_{2m(k)-1}, Sx_{2n(k)}, a)]$$
$$\le V_{2n(k)} + \phi[d(x_{2n(k)}, x_{2m(k)-1}, a),\ V_{2n(k)}, V_{2n(k)-1}, d(x_{2n(k)}, x_{2m(k)}, a), d(x_{2m(k)-1}, x_{2n(k)+1}, a)]$$

As ϕ is upper semi continuous from the right, we obtain

$$\le \phi(\in, 0, 0, \in, \in) < \in, \text{ when } k \to \infty,$$

which is a contradiction. Thus $\{x_n\}$ is a Cauchy sequence. By the completeness of X, $\{x_n\}$ converges to a point $z \in X$.

Now

$$d(z, Sz, a) \le d(z, x_{2n}, a) + d(z, Sz, x_{2n}) + d(Sz, Tx_{2n-1}, a)$$
$$\le d(z, x_{2n}, a) + d(z, Sz, x_{2n}) + \phi[d(z, x_{2n-1}, a), d(z, Sz, a),\ d(x_{2n-1}, Tx_{2n-1}, a), d(z, Tx_{2n-1}, a), d(x_{2n-1}, Sz, a)]$$
$$\le d(z, Sz, x_{2n}) + d(z, x_{2n}, a) + \phi[d(z, x_{2n-1}, a), d(z, Sz, a), V_{2n-1}, d(z, x_{2n-1}, a) + d(x_{2n-1}, Tx_{2n-1}, a) + d(z, Tx_{2n-1}, x_{2n-1})\, d(x_{2n-1}, Sz, a)]$$

Letting n tending to ∞, we obtain

$$d(z, Sz, a) \le \phi(0, d(z, Sz, a), 0, 0, d(z, Sz, a)$$

By the property of ϕ, we get $d(z, Sz, a) = 0$. Similarly, it can be shown that $d(z, Tz, a) = 0$. Thus $z \in Sz \cap Tz$. This completes the proof.

Remark 1. Taking to be continuous function for metric space we get a slightly revised version of the result of Husain and Sehgal (1975) as a corollary.

References

Cho Y J, Khan M S, Singh S L 1988 Common fixed points of weakly commuting mappings. *Prirod-Mat Fak Ser Mat* **18**(1) 129-142.

Gahler S 1963/64 2-Metrische Raume un topologische. *Struktur Math Nachr* **62** 115-148.

Husain S A, Sehgal V M 1975 On common fixed points for a family of mappings. *Bull Austral Math Soc* **13** 344-348.

Matkowski J 1977 Fixed point theorems for mapping with contractive iterate at a point. *Proc Amer Math Soc* **62** 344-348.

Singh S P, Meade B A 1977 On common fixed point theorems. *Bull Austral Math Soc* **16** 49-53.